On a lemma of Łojasiewicz

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Abstract

The purpose of this short note is to make better known (and prove a slightly extended version of) a lemma on the boundary behavior of analytic functions. This lemma seems to have been proved for the first time by Stanisław Łojasiewicz in his paper (possibly the very first published one) [4] on the standard Fatou-Riesz theorem.

1 Basic notions

Definition 1. *Let $A, B \subset \mathbb{C}$ two subsets in the complex plane: their* (Euclidean) distance $\rho : \mathscr{P}(\mathbb{C}) \times \mathscr{P}(\mathbb{C}) \to \mathbb{R}_{\geq 0}$ *is defined as*

$$\rho(A, B) = \inf_{z_1 \in A \wedge z_2 \in B} |z_1 - z_2|$$

In the sequel we will deal only with distances between a point z inside a domain G and its boundary ∂G or the distance between two points $z_1, z_2 \in G$: thus, by abuse of notation, we respectively write $\rho(\{z\}, \partial G) \triangleq \rho_{\partial G}(z)$ and $\rho(\{z_1\}, \{z_2\}) \triangleq \rho(z_1, z_2)$.

Definition 2. *Let $G \in \mathbb{C}$ be a domain, $\zeta_0 \in \partial G$ a point on its boundary, $M > 1$, $0 < \varepsilon \leq 1$ and $\theta \in [0, 2\pi]$ three real numbers. A θ-*directed (ε, M)-conical region with vertex ζ_0 *is the conical set $\Gamma^\varepsilon_\theta(\zeta_0, M) \subset \mathbb{C}$ defined as*

$$\Gamma^\varepsilon_\theta(\zeta_0, M) = \left\{ z \in \mathbb{C} : \big(|z - \zeta_0| < \varepsilon\big) \wedge \left(0 \leq |\arg(\zeta_0 - z) - \theta| \leq \arccos \tfrac{1}{M}\right) \right\} \quad (1)$$

such that its closure is included in the closure of G, i.e. $\overline{\Gamma^\varepsilon_\theta(\zeta_0, M)} \subset \overline{G}$. The point ζ_0 is the vertex *of the conical region while ε, θ and M are respectively its* magnitude, direction *and* aperture. *When the vertex and the aperture are clearly and unambiguously inferred from the context or are not relevant for the discourse, the notation for the conical region will be safely abridged as $\Gamma^\varepsilon_\theta$.*

The following definition is inspired by the one given in [1, §1.1, p. 8].

Definition 3. *Let $f : G \to \mathbb{C}$ be a holomorphic function and $\zeta_0 \in \partial G$ a point on the boundary of its domain of definition: f is said to have a* non-tangential limit *as $z \to \zeta_0$ if and only if there exists a conical region $\Gamma^\varepsilon_\theta(\zeta_0, M)$ such that*

$$\lim_{t \to 1} f(\gamma(t)) = s$$

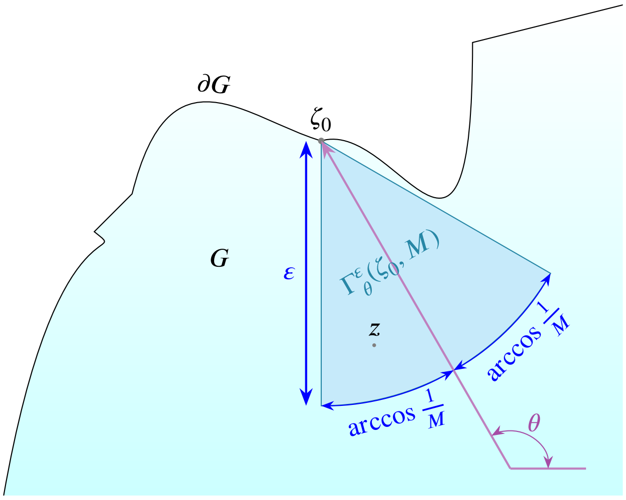

Figure 1: Graphical glossary for definition 2.

along all continuous curves $\gamma : [0,1] \to \mathbb{C}$ *such that*

$$\begin{cases} \gamma(1) = \zeta_0, \\ \gamma([0,1]) \subseteq \Gamma_\theta^\varepsilon(\zeta_0, M). \end{cases}$$

The region $\Gamma_\theta^\varepsilon(\zeta_0, M)$ *is then called a* conical approach region for f.

2 The fundamental lemma an its generalization

The following lemma was proved by Stanisław Łojasiewicz: the exposition closely follows the original one in its strength and simplicity.

Lemma 1. [4, lemme II, p. 242] *Let* $G \subset \mathbb{C}$ *be a domain,* $\zeta_0 \in \partial G$ *a point on its boundary,* $f(z)$ *a holomorphic function on* G *such that its limit for* $z \to \zeta_0$ *exists and finally let* $\rho_{\partial G}(z)$ *be the distance between* z *and* ∂G*: then*

$$\left(z - \zeta_0\right) \cdot f'(z) \underset{z \to \zeta_0}{\longrightarrow} 0$$

as long as the quotient $\left|\zeta_0 - z\right| / \rho_{\partial G}(z)$ *is bounded.*

Proof. Let $s = \lim_{z \to \zeta_0} f(z)$, $C_z = \left\{\zeta \in G : |\zeta - z| = \frac{\rho_{\partial G}(z)}{2}\right\}$ and $\eta(z) = \max_{C_z} |f(z) - s|$. For all $z \in G$,

$$f'(z) = \frac{1}{2\pi i} \int_{C_z} \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta = \frac{1}{2\pi i} \int_{C_z} \frac{f(\zeta) - s}{(\zeta - z)^2} \mathrm{d}\zeta,$$

thus it follows that

$$\left|f'(z)\right| \leqslant \frac{1}{2\pi} \pi \rho_{\partial G}(z) \frac{\eta(z)}{\left(\frac{1}{2}\rho_{\partial G}(z)\right)^2} = 2\frac{\eta(z)}{\rho_{\partial G}(z)}$$

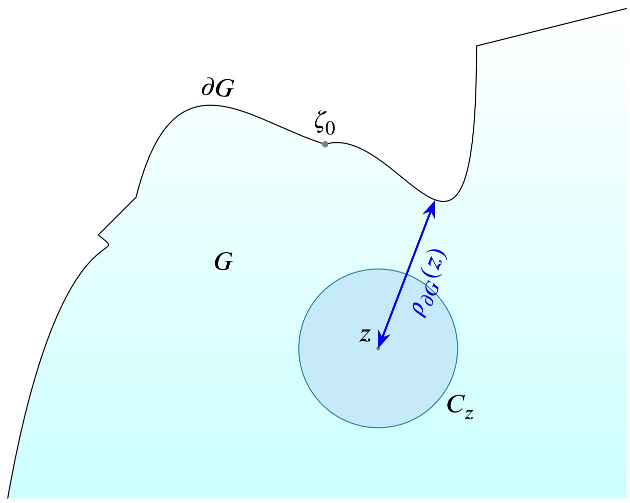

Figure 2: Graphical representation for Łojasiewicz's lemma 1.

and consequently

$$\left|\left(z-\zeta_0\right)\cdot f'(z)\right| \leqslant 2\frac{\left|z-\zeta_0\right|}{\rho_{\partial G}(z)}\eta(z)$$

which implies the thesis since $\eta(z)\to 0$ for $z\to\zeta_0$. □

The proof of the following lemma is formally the same given by Łojasiewicz for lemma 1: nevertheless, a careful choice of the objects and concepts involved, precisely the consideration of the approach region for f and the requirement on the existence of a nontangential limit instead of a "tout court" limit avoids the necessity of further technicalities and moreover shed some light on its meaning.

Lemma 2. *Let $G\subset\mathbb{C}$ be a domain, $\zeta_0\in\partial G$ a point on its boundary, $f(z)$ a holomorphic function on G such that its non tangential limit for $z\to\zeta_0$ exists and finally let $\rho_{\partial\Gamma_\theta^\varepsilon}(z)$ be the distance of z from the boundary $\partial\Gamma_\theta^\varepsilon\left(\zeta_0,M\right)$ of the conical approach region for f: then*

$$\left(z-\zeta_0\right)\cdot f'(z) \underset{z\to\zeta_0}{\longrightarrow} 0$$

as long as the quotient $\left|\zeta_0-z\right|/\rho_{\partial\Gamma_\theta^\varepsilon}(z)$ is bounded.

Proof. Let $s=\lim\limits_{z\to\zeta_0}f(z)$, $C_z=\left\{\zeta\in G:|\zeta-z|=\frac{\rho_{\partial\Gamma_\theta^\varepsilon}(z)}{2}\right\}$ and $\eta(z)=\max\limits_{C_z}|f(z)-s|$. For all $z\in G$,

$$f'(z)=\frac{1}{2\pi i}\int\limits_{C_z}\frac{f(\zeta)}{(\zeta-z)^2}\,\mathrm{d}\zeta=\frac{1}{2\pi i}\int\limits_{C_z}\frac{f(\zeta)-s}{(\zeta-z)^2}\,\mathrm{d}\zeta,$$

thus it follows that

$$\left|f'(z)\right|\leqslant\frac{1}{2\pi}\pi\rho_{\partial\Gamma_\theta^\varepsilon}(z)\frac{\eta(z)}{\left(\frac{1}{2}\rho_{\partial\Gamma_\theta^\varepsilon}(z)\right)^2}=2\frac{\eta(z)}{\rho_{\partial\Gamma_\theta^\varepsilon}(z)}$$

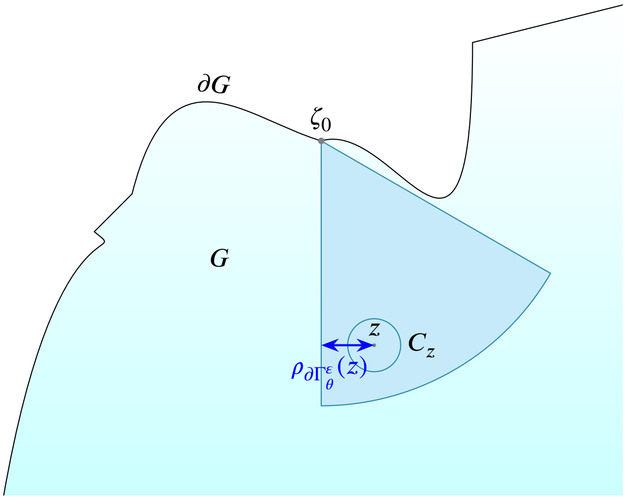

Figure 3: Graphical representation for lemma 2.

and consequently

$$\left|\left(z-\zeta_0\right)\cdot f'(z)\right| \leqslant 2\frac{\left|z-\zeta_0\right|}{\rho_{\partial\Gamma_\theta^\varepsilon}(z)}\eta(z)$$

which implies the thesis since $\eta(z)\to 0$ for $z\to\zeta_0$. □

Remark 1. *It can be easily checked that the region where conclusion of lemma 2 holds i.e. the quantity* $\left|\zeta_0 - z\right|/\rho_{\partial\Gamma_\theta^\varepsilon}(z)$ *is bounded, is any conical region* $\Gamma_\theta^\varepsilon\big(\zeta_0, \widehat{M}\big)$ strictly contained *in the conical approach region for* f *iff* $\hat{M} < M$. *However I kept the same notation used by Lojasiewicz in order to emphasize the absolute analogy between the two results.*

Remark 2. *Lemma 2 generalizes lemma 1 since it holds for some classes of functions* f *for which* $\lim_{z\to\zeta_0} f(z)$ *does not exists and therefore the latter does not apply. To see this, consider the holomorphic function* $f:\mathbb{D}\to\mathbb{C}$ *defined as*

$$f(z) = e^{-\left(\frac{1+z}{1-z}\right)^5},$$

and evaluate its limits as $z\to 1$ *along different paths. It can be seen that* $\lim_{z\to 1} f(z)$ *does not exists thus lemma 1 does not hold: nevertheless, since* $f(z)$ *goes* 0 *along any path contained in the conical region* $\Gamma_0^\varepsilon\left(1, 1/\cos\frac{\pi}{5}\right)$ *lemma 2 can be applied.*

3 Final notes and observations

This lemma clarifies a doubt that I expressed in a MathOverflow question [3]. Precisely, in his work [2, §1, formula 2.2, p. 234] on the estimation of the remainder of a power series $f(z)$ at a boundary point (assumed to be $z=1$) of the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, Giovanni Ricci assumes for the first derivative of f the following asymptotic behavior.

$$|f'(z)| < K\left|z-e^{i\theta}\right|^{\alpha-1} \quad z\in\mathbb{D}\cap\left\{z\in\mathbb{C}:|z-e^{i\theta}|<\rho\right\}$$

where $0 < \alpha \leq 1$, $\rho > 0$ and $K > 0$. I questioned it as somewhat artificial, but in the light of lemma 1 this choice appears quite natural: Ricci seems to show how his general approach works in an explicit instance of the general behavior.

References

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[3] D. Tampieri, On the remainder of a power series evaluated on the boundary of its convergence disk, MathOverflow. Available at https://mathoverflow.net/q/396814.

[4] S. Łojasiewicz, Une demonstration du théorème de Fatou, *Annales de la Société Polonaise de Mathématique* **22** (1950), 241–244 (French). MR 0038429. Zbl 0035.33901. Available at https://rcin.org.pl/publication/44441.

Revision History

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1.0	05.12.2022	Daniele Tampieri	Creation
1.1	19.05.2023	Daniele Tampieri	First published version
1.2	22.05.2023	Daniele Tampieri	Typo fixes
1.3	29.05.2023	Daniele Tampieri	Added pictures and clarified some points